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A two-pion exchange three-nucleon force based on a realistic πN interaction

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Abstract. The contribution of a $\pi\pi$-exchange three-body force to the three-nucleon binding energy is calculated in terms of a πN amplitude. The latter is based on a meson-theoretical model of πN interaction developed by the Jülich group. Similar to a previous study based on simple phenomenological πN potentials a very small effect of the resulting three-body force is found. Possible origins of the two-orders-of-magnitude discrepancy between the present result and the values obtained for the Tucson-Melbourne three-body force are investigated. Evidence is provided that this discrepancy is most likely due to strikingly different off-shell properties of the πN amplitudes underlying the two approaches.

PACS. 21.10.Dr Binding energies and masses – 21.45.+v Few-body systems – 25.80.Dj Pion elastic scattering

1 Introduction

Recently T.-Y. Saito and I. R. Afnan (SA) [1,2] have calculated the contribution of a $\pi\pi$-exchange three-body force (see Fig. 1) to the three-nucleon binding energy in terms of the πN amplitude using perturbation theory. Their approach determines the contributions of the different πN partial waves and (via the division of the πN amplitude into a pole and nonpole term) allows for a consistent determination of the πNN form factor. The calculations are based on phenomenological separable πN potentials [3]. The total contribution of this three-body force (TBF) to the binding energy of the triton has been found to be very small. It is typically of the order of a few keV. This result falls short of calculations based on the Tucson-Melbourne (TM) and the Brazilian $\pi\pi$-exchange three-nucleon potentials [4–6]. The latter potentials make a contribution to the binding energy of the triton that is of the order of the discrepancy between experiment and calculations with realistic nucleon-nucleon potentials (i. e. about 1 MeV).

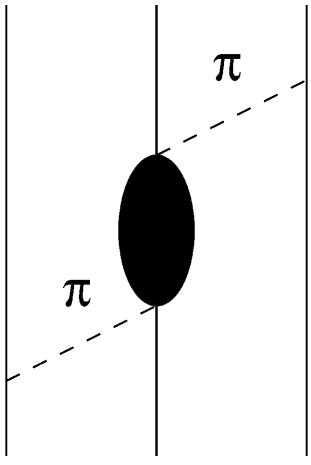

Fig. 1. The contribution to the three-nucleon force

The origin of this surprisingly large difference of two orders of magnitude in the contribution of the three-body force has been the topic of two subsequent very detailed investigations by Saito and Afnan [2] and Murphy and Coon [7]. SA concluded that the total contribution of the TBF to the binding energy of the triton found in their approach is so small as a result of the energy dependence of the πN amplitudes, cancellations between the contributions from the S- and P-wave πN partial waves, and in particular, the soft πNN form factor. Indeed the form factors extracted from their πN interaction models correspond to monopole cutoff masses of around or even less than 0.4 GeV - which have to be compared to values of about 0.7 - 0.8 GeV suggested from other information [9–11] and to the value of 0.8 GeV used in calculations with the TM potential.

Murphy and Coon carried out a thorough comparison of the πN amplitudes that underlie the TM TBF and the calculations of SA [7]. They criticized that the amplitudes used by SA do not fulfil the low-energy theorems of the πN interaction as imposed by chiral symmetry. But still they attested that these amplitudes are qualitatively simi-

lar to the one used in the TM potential. The prime reason for the two-orders-of-magnitude discrepancy found in the contributions of the TBF was suspected to be likewise the soft πNN form factor emerging from the phenomenological separable πN interactions employed by SA.

In the present paper we want to re-investigate the origin of this large discrepancy in the predicted contribution of the TBF. We follow the same approach as SA. However, we start out from a meson-theoretical πN model developed recently by the Jülich group [12]. This model, besides being conceptionally much better founded than the simple separable potentials employed in [1,2], has the important advantage that it does not exhibit those deficiencies which led to a criticism on the work by Saito and Afnan. Firstly, the Jülich model is in agreement with empirical information on the πN amplitude in the subthreshold region [13]. In particular its prediction for the amplitude at the so-called Cheng-Dashen point is close to the empirical value. Second, and most importantly it yields πNN form factors which are comparable to those of the TM potential, with monopole cutoff masses of around 0.7 GeV [14]. Therefore this model provides an ideal starting point for re-assessing the role of the $\pi\pi$-exchange TBF in the binding of the three-nucleon system.

The paper is structured in the following way: In Sect. 2 we review the salient features of the Jülich πN model. In particular we concentrate on those properties that are relevant for the present study. The specific structure of our three-nucleon code makes it necessary to represent the meson-theoretical πN amplitude in separable form. For this purpose we applied the so-called EST method [15] which allows to generate separable representations that agree exactly (on- and half-off-shell) with the original interaction at specific predetermined energies. This method is briefly described also in Sect. 2. Furthermore, we discuss the reliability of the separable representation by comparing (off-shell) amplitudes obtained from it to the ones of the original Jülich model for various πN partial waves.

The formulation of the TBF is given in Sect. 3 together with a short outline of the formalism. Results for the contribution of the $\pi\pi$-exchange TBF to the triton binding energy, based on the πN amplitude of the Jülich model are presented in Sect. 4. Anticipating our results we find again that the TBF is very small. Therefore, in Sect. 5, we embark on a detailed discussion of the πN amplitudes on which the TM TBF and our calculation are based. We focus specifically on the off-shell properties of these amplitudes since they are determined quite differently in the two approaches. Indeed, we will argue that much of the observed two-orders-of-magnitude discrepancy in the binding energy is due to off-shell effects and we will present numerical evidence for this conjecture. Finally, a summary is given in Sect. 6.

2 The πN interaction

The πN models employed in the present investigation are based on meson exchange and have been developed by the Jülich Group [12]. They include the s-channel and u-channel nucleon and delta-isobar pole diagrams together with correlated $\pi\pi$ exchange in the $J^P = 0^+$ (σ) and 1^- (ρ) channels as shown in Fig. 2. The interaction potentials are derived in time ordered perturbation theory and then unitarized by means of a relativistic (Lippmann-Schwinger type) scattering equation

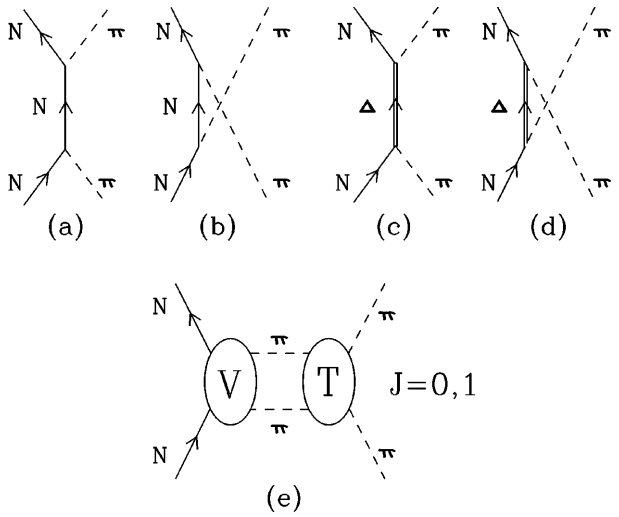

Fig. 2. Diagrams included in the πN potential

$$T = V + VG_0T \, . \tag{1}$$

The resulting models account for the scattering data in the elastic region as well as for the low-energy parameters [12]. Furthermore they also satisfy chiral symmetry constraints. In particular, the resulting values for the so-called πN Σ term (the contribution of the isoscalar forward scattering amplitude at the Cheng-Dashen point) - $\Sigma = 66.4$ (65.6) MeV for model 1 (2) of [12] - are in good agreement with the empirical value of $\Sigma = 60$ MeV [16].

Since in the discussion of Murphy and Coon [7] special attention was given to the subthreshold behaviour of the πN amplitudes, we want to present here the corresponding results for the Jülich πN model. In the continuation of the T-matrix for the potential models to the subthreshold region, we follow the procedure outlined in section 4 of [7]. We calculated the (on-shell) background isoscalar amplitude $\bar{F}^+(\nu, t)$ (conventionally called $\bar{D}^+$; for definition see, e.g., [7,8]) from the subthreshold point $\nu = 0$, $t = 0$ to the Cheng-Dashen point ($\nu = 0$, $t = 2m_\pi^2$). The predictions of the Jülich models 1 and 2 are shown in Fig. 3 in comparison to the ones of the πN amplitudes that form the basis of the TM and Brazil $\pi\pi$-exchange TBF. One can see that the results of the meson-exchange models more or less coincide with the πN amplitude employed in the TM TBF. Furthermore, they are also in rather nice agreement with the empirical subthreshold amplitude given by Höhler [8].

In a recent paper C. Schütz et al. [14] have determined the πNN vertex functions resulting from the models of [12] (cf. [14] and the Appendix for definitions and relevant formulue). From the vertex functions at the nucleon pole, the decrease in the πNN form factors $F_{\pi NN}(p^2)$

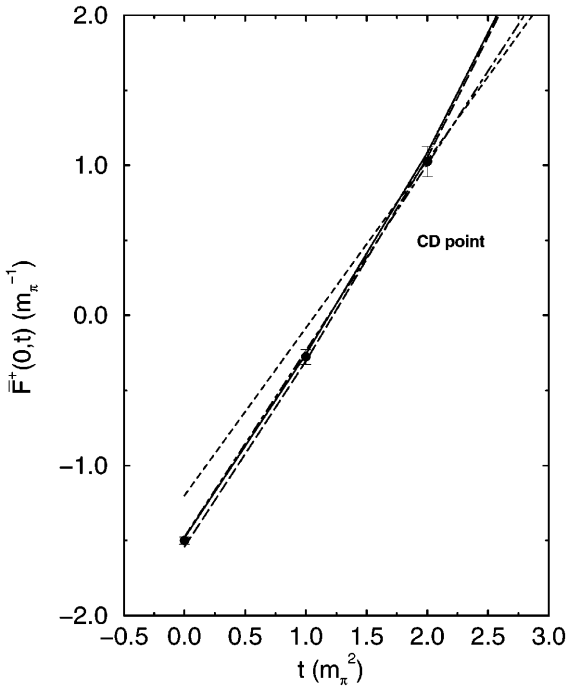

Fig. 3. Comparison of the background isoscalar πN amplitude $\bar{F}^+(0,t)$. The solid (long-dashed) lines are obtained from the Jülich πN model 1 (2) of [12]. The dash-dotted (short-dashed) lines are the predictions of the amplitudes that form the basis of the Tucson-Melbourne (Brazil) $\pi\pi$-exchange three-body force. Empirical values at the Cheng-Dashen (CD) point, $t = m_\pi^2$ and $t = 0$ are taken from [8]

from the pion pole $p^2 = m_\pi^2$ to $p^2 = 0$, has been extracted following the procedure proposed by Mizutani et al. [17]. This quantity is a measure for the softness of the πNN form factor. It has been found that the πNN form factors implied by the models considered in [14] are, in general, significantly harder than the ones used by SA in their study of the contribution of the $\pi\pi$-exchange three-nucleon force to the $3N$ binding energy. In particular, the model 2' of [14] yields a decrease in the form factor from the pion pole to $p^2 = 0$ of slightly less then 4% - which is quite close to the value of 3% implied by the form factor introduced in the TM three-nucleon force and also in good agreement with a recent lattice QCD calculation [9] and other independent information [10,11]. Accordingly, the major concern raised by Murphy and Coon against the work of SA does not apply for this model and therefore it provides an excellent starting point for re-analyzing the contributions of the $\pi\pi$-exchange three-nucleon force to the $3N$ binding energy in the approach of Saito and Afnan.

In this work we will also present results for the other models considered in [14] and it is appropriate to say some words about the basic differences between these models. All the models are based on the same dynamical input (cf. Fig. 2). They differ, however, in the (phenomenological) parametrization of the (bare) vertex form factors. Details and explicit formulae can be found in Sect. 3 of [12]. Both models provide a similarly good description of the πN scattering data. However, the differences in the parametrization of the vertex form factors lead to different vertex functions and in turn to different (dressed) πNN form factors.

Two further models (1' and 2') have been presented in [14] for the following reason: The models 1 and 2 are constructed by assuming πNN pseudovector (pv) coupling. However, the πNN form factor derived from lattice QCD calculations [9], and the values of $F_{\pi NN}(0)$ from other independent information [10,11] (and also the values given for the models used by SA), are based on pseudoscalar (ps) coupling. In order to allow for a meaningful comparison for the different couplings the authors of [14] have constructed variants of models 1 and 2 (labelled 1' and 2') where ps coupling is used in the nucleon s-channel pole term, cf. Fig. 2(a).

Values for $F_{\pi NN}(0)$ for the various models are compiled in Table 1. (Note that $F_{\pi NN}$ is normalized to $F_{\pi NN}(m_\pi^2) = 1$.) For the ease of comparison we also include here the result for the separable πN model PJ by McLeod and Afnan [3] which has been used (amongst others) in the investigation by SA and on which most of the concerns and criticism of Murphy and Coon are based. It is evident that for this model the decrease of the form factor from the pion pole to $p^2 = 0$ is much larger - almost 20%. We want to emphasize here, however, that it is the πNN vertex function which enters into the calculation of the TBF and not the form factor (cf. Sect. 3). Therefore we also show this quantity (cf. Fig. 4). Note that the πNN vertex functions for models 1 and 1' are practically the same. Accordingly we expect these models to give the same result for the contributions of the $\pi\pi$-exchange TBF to the $3N$ binding energy, and therefore we will consider only model 1 in the following analysis. Furthermore, the vertex function for model 2 is obviously harder than the one resulting from model 2' (for small and intermediate momenta) - in contrast the form factors for model 2' looks harder (cf. Table 1). This seemingly paradoxical situation has been discussed thoroughly in [14].

Before carrying out the actual three-nucleon calculations the meson-theoretical πN models have to be expanded in separable form. This is necessitated by the specific structure of our three-nucleon code which can only deal with interaction models given in separable form. For this purpose we apply the so-called Ernst-Shakin-Thaler (EST) method [15] which allows us to generate separable representations of arbitrary rank N that agree exactly (on- and half-off-shell) with the original reaction matrix at N specific predetermined energies.

Let us begin with the (partial wave projected) Lippmann-Schwinger equation for the radial wave function

$$|\psi_E\rangle = |k_E\rangle + G_0(E)V|\psi_E\rangle, \tag{2}$$

where $|k_E\rangle$ is the incoming wave and $G_0(E)$ the two-body Green's function. (In (2) and in the following the partial wave index is suppressed for convenience.) For proper

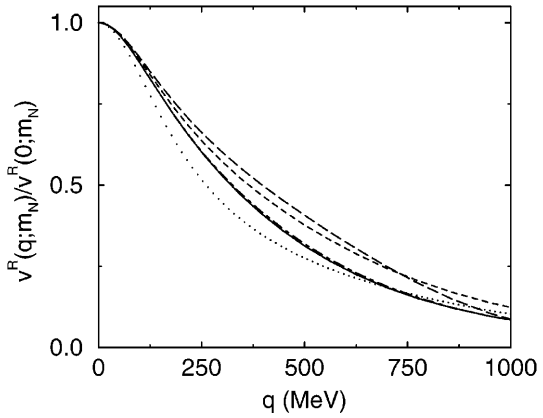

Fig. 4. πNN vertex functions as function of the pion momentum in the πN c.m. system. The solid (long-dashed) line denotes the prediction resulting from model 1 (2) of [12], the dash-dotted (short-dashed) line denotes the prediction of model 1' (2') of [14], the dotted line shows the result of model PJ of [3]

scattering solutions (on-shell) k_E and E are related by

$$E = \sqrt{m_N^2 + k_E^2} + \sqrt{m_\pi^2 + k_E^2}. \tag{3}$$

According to the EST method a rank-N separable representation for the potential V is given by

$$\tilde{V} = \sum_{i,j=1}^{N} V|\psi_{E_i}\rangle \lambda_{ij} \langle\psi_{E_j}|V \ , \tag{4}$$

where E_i, $(i = 1, ..., N)$, are N freely chosen energies. The coupling strengths λ_{ij} are determined by the condition

$$\sum_{j=1}^{N} \lambda_{ij} \langle\psi_{E_j}|V|\psi_{E_k}\rangle = \delta_{ik} \ . \tag{5}$$

It is evident from (4) that the "form factors" of the separable potential $\tilde{V}$ [18] consist of the objects $V|\psi_{E_i}\rangle$, where $|\psi_{E_i}\rangle$ are solutions of (2) for the potential V at the energies E_i. Therefore, by virtue of (5), the following relation holds

$$\tilde{V}|\psi_{E_i}\rangle = V|\psi_{E_i}\rangle = T(E_i)|k_{E_i}\rangle = \tilde{T}(E_i)|k_{E_i}\rangle \tag{6}$$

at the N energies E_i, where $\tilde{T}$ is the solution of the Lippmann-Schwinger equation (1) for the separable representation $\tilde{V}$. This means that the on-shell as well as the half-off-shell t-matrix for both interactions V and $\tilde{V}$ are exactly the same at the energies E_i.

In the present case the interaction models V are energy-dependent and therefore a modification of this scheme proposed by B. Pearce [19] is employed. According to it the condition

$$\begin{aligned} \langle\psi_{E_l}|V(E)|\psi_{E_k}\rangle &= \langle\psi_{E_l}|\tilde{V}(E)|\psi_{E_k}\rangle \\ &= \sum_{i,j=1}^{N} \langle\psi_{E_l}|V(E_i)|\psi_{E_i}\rangle \lambda_{ij}(E) \\ &\quad \cdot \langle\psi_{E_j}|V(E_j)|\psi_{E_k}\rangle \end{aligned} \tag{7}$$

has to be used for determining the coupling strengths $\lambda_{ij}(E)$ instead of (5). As expected also the separable representation becomes now energy-dependent.

A special treatment is required for the P_{11} partial wave which contains the (s-channel) nucleon pole. It must be possible to clearly separate the contribution of this pole term from the total P_{11} amplitude. Its contribution to the three-nucleon binding energy is already taken into account by solving the standard bound-state Faddeev equations. Therefore, in order to avoid double counting, only the non-pole part of the P_{11} must be considered in the present investigation. Furthermore in the consistent approach of SA the πNN vertex function is extracted from this pole term and is then used for the vertices where the pions are emitted (absorbed) by (at) the outer nucleons. Consequently a separable representation for the P_{11} partial wave must guarantee that (a) the non-pole amplitude is reliably reproduced and (b) the πNN vertex function extracted from the pole term agrees exactly with the one obtained for the original interaction model. This can be achieved and we summarize details of the construction procedure in the Appendix.

Of course if one relies on such separable representations one has to ensure that they incorporate all the relevant properties of the original interaction models. From extensive tests, we find that a rank-1 separable representation is sufficient for the present purpose provided that the expansion energy is choosen in the relevant energy domain (i. e. around the triton binding energy which corresponds to a center-of-mass energy of roughly 930 MeV in the πN system). Thus we have selected this particular energy for the separable representation to be applied in the present study. Since this energy is below the elastic πN threshold, k_E in (2) can no longer be fixed by the on-shell condition. Following our previous work [20] we choose k_E in such a way that ik_E fulfils (3). For $E_1 = 930$ MeV this implies $k_{E_1} \approx 138$ MeV/c.

The quality of the separable represention is demonstrated in the Figs. 5 and 6 for model 2. Figure 5 shows the off-shell transition amplitude $T_\alpha(q, q'; Z)$ for fixed off-shell momenta $q = q' = 130$ MeV as a function of the total energy Z. Note that in case of the P_{11} partial wave only the non-pole part of the t-matrix is shown since, as explained above, this is the part relevant for the present study. Figure 6 shows the transition amplitude $T_\alpha(q, q'; Z)$ for fixed Z and q' as a function of the other off-shell momentum. In the latter figure we display only results for those partial waves that are expected to give the dominant contribution of the $\pi\pi$-exchange three-nucleon force to the $3N$ binding energy [1], namely S_{31}, P_{11}, and P_{33}. We want to point out, however, that the agreement in the other partial waves is of similar quality. Likewise we want to refrain from displaying corresponding results for the model 1 here since the quality of its separable representation is pretty much the same. Furthermore, we do not show the πNN vertex functions resulting from the separable representation because - by construction - they are identical to the ones of the original model (cf. the Appendix).

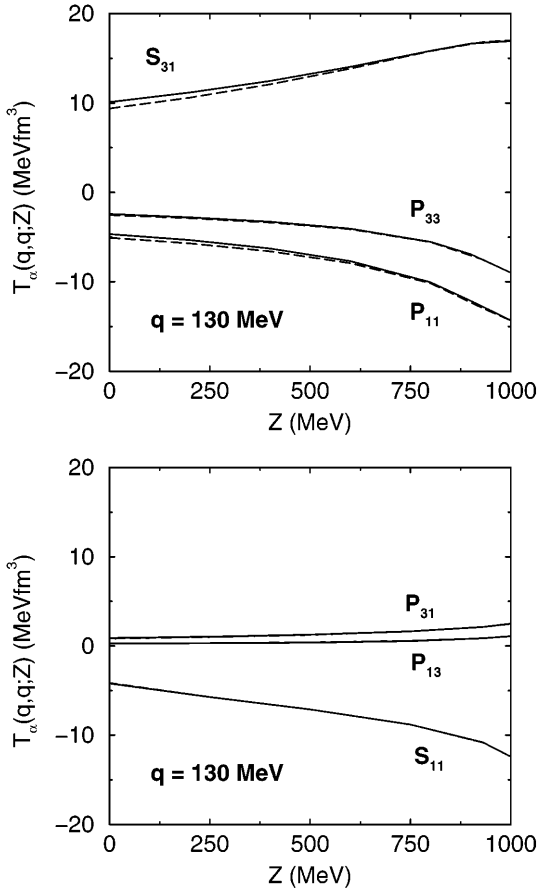

Fig. 5. Off-shell πN transition amplitude $T_\alpha(q, q'; Z)$ for $q = q' = 130$ MeV/c as a function of the total energy Z. The full line is the result of model 2' of [14] whereas the dashed line corresponds to the rank-1 separable representation described in the text

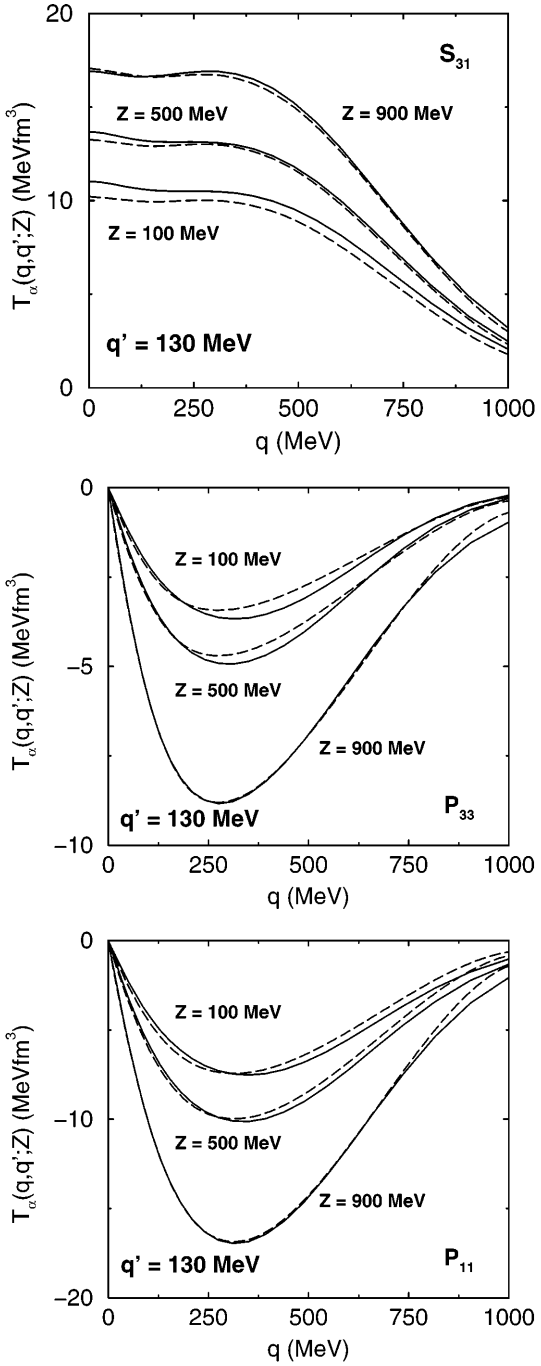

Fig. 6. Off-shell πN transition amplitude $T_\alpha(q, q'; Z)$ for $q' = 130$ MeV/c at three different energies Z as a function of the off-shell momentum q. Same description as in Fig. 5

Note that we have also constructed rank-1 separable representations where the expansion energies are near the πN threshold ($E_1 = 1077$ MeV for S_{11}; $E_1 = 1000$ MeV for S_{31}; $E_1 = 1100$ MeV for P_{11}, P_{31}, P_{13}, P_{33}). (In case of the S_{31} partial wave the amplitude around 1077 MeV has a peculiar energy dependence which would require, in principle, a rank-2 representation. In order to avoid this complication we have chosen a somewhat lower value for E_1.) These representations will be also employed in our investigations. They will serve as a term of reference for how strongly the resulting $3N$ binding energies depend on the specific choice of the separable representation.

3 Formulation of the three-body force

In this section we formulate the TBF using the πN amplitude and πNN vertex function which have been derived in the previous section.

We follow the prescription by SA. The TBF is schematically shown in Fig. 7, i.e., (i) a pion is emitted from the first nucleon, (ii) the pion is scattered off the second nucleon, (iii) the pion is absorbed by the third nucleon. The strength function of pion emission and absorption is given by the πNN vertex function which depends on the energy of the πN system as explained in the previous section. Also the πN amplitude corresponding to the scattering of the pion on the second nucleon is energy dependent and includes only the non-pole contribution in the P_{11} channel. We introduce Jacobi variables in the πNNN system so that we can define the πN relative momenta and energies at the stages (i), (ii) and (iii).

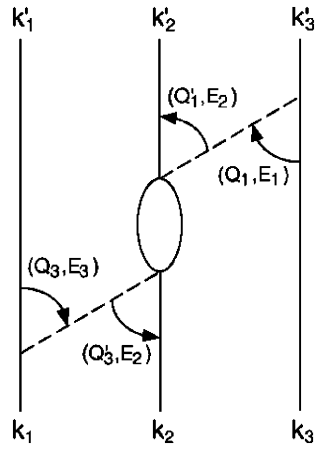

Fig. 7. Illustration of the three-body force. See the text for the definitions of functions, momenta and energies

The momenta of the three nucleons before the three-body interaction in Fig. 7 are given by $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ and the momenta after the interaction are $\mathbf{k}'_1$, $\mathbf{k}'_2$ and $\mathbf{k}'_3$, respectively. We define the relative momenta between the third nucleon and the system of first and second nucleon and pion ($\mathbf{q}_3$), between the second nucleon and the system of first nucleon and pion ($\mathbf{p}_3$) and between the first nucleon and pion ($\mathbf{Q}_3$) at the stage (i)

$$\mathbf{q}_3 = -\mathbf{k}_3, \tag{8}$$

$$\mathbf{p}_3 = \frac{m_N(\mathbf{k}_\pi + \mathbf{k}'_1) - (m_N + m_\pi)\mathbf{k}_2}{(2m_N + m_\pi)}, \tag{9}$$

$$\mathbf{Q}_3 = \frac{m_N\mathbf{k}_\pi - m_\pi\mathbf{k}'_1}{(m_N + m_\pi)}, \tag{10}$$

where $\mathbf{k}_\pi$ is the pion momentum. Then the center-of-mass energy in the system of first nucleon and pion, E_3, is obtained using those relative momenta

$$E_3 = E + m_N - \frac{q_3^2}{2\mu_2} - \frac{p_3^2}{2\mu_1} \tag{11}$$

where $E = -E_T$ is the total energy of the whole system not including rest masses, and the reduced masses μ_1 and μ_2 are defined, respectively, by the relations

$$\frac{1}{\mu_1} = \frac{1}{m_N} + \frac{1}{m_N + m_\pi}, \tag{12}$$

and

$$\frac{1}{\mu_2} = \frac{1}{m_N} + \frac{1}{2m_N + m_\pi}. \tag{13}$$

In the same way we define the relative momenta and energies of the system of second nucleon and pion, $\mathbf{Q}'_3$ (before scattering), $\mathbf{Q}'_1$ (after scattering) and E_2 at the stage (ii) and $\mathbf{Q}_1$ and E_1 of the system of third nucleon and pion at the stage (iii), respectively.

Using these πN relative momenta and energies the TBF, $W(E)$, can be symbolically written as

$$W(E) = v^R_{\pi N}(Q_1; E_1)\; G_{\pi NNN}(E)\; T_{\pi N}(Q'_1, Q'_3; E_2) \cdot G_{\pi NNN}(E)\; v^R_{\pi N}(Q_3; E_3). \tag{14}$$

Here, $v^R_{\pi N}(Q_i; E_i)$ ($i = 1, 3$) is the renormalized πNN vertex function (cf. the Appendix) which gives the strength of the pion emission and absorption on the nucleon. $G_{\pi NNN}(E)$ is the propagator of the πNNN system and $T_{\pi N}(Q'_1, Q'_3; E_2)$ is the non-pole part of the πN scattering amplitude. (14) represents how the pion is emitted from the first nucleon, scattered off the second nucleon, and then absorbed by the third nucleon. Note that the πN amplitude and the πNN vertex function are determined in the same framework, i. e. they are obtained from the same πN interaction model. A detailed discussion of (14) can be found in [2]. Since in the Jülich model the energy is defined fully relativistic whereas a semi-relativistic form is employed by SA, we have to change the πNNN propagator of [2] to be

$$G_{\pi NNN}(E) = \left(E_i - \sqrt{Q_i^2 + m_N^2} - \sqrt{Q_i^2 + m_\pi^2}\right)^{-1}, \tag{15}$$

with $i = 1$ or 3 (cf. (4.9) of [2]).

4 Results

The contribution of the TBF to the binding energy of the three-nucleon system, $\Delta E^{(3)}$, is calculated in first order perturbation theory, i. e.

$$\Delta E^{(3)} = \langle\Psi|W(-E_T)|\Psi\rangle, \tag{16}$$

where $|\Psi\rangle$ is the triton wave function. This wave function is obtained from solving the Faddeev equations for the so-called PEST potential [20,21] which is a separable representation of the Paris NN potential [23] derived by the EST method. All nucleon-nucleon partial waves with total angular momentum less than or equal two are employed in the calculation. The triton properties obtained by the PEST potential are comparable with those by the original Paris potential as shown in [22,2].

Evidently, like in the work by SA there is no consistency between the NN interaction which is used to generate the triton wave function and the TBF. However, we are making use of the Born approximation and therefore the triton wave function and the TBF are obtained separately, anyway. Thus, as argued already by SA [2], we do not expect that this inconsistency has an influence on the qualitative features of our results. One should also keep in mind that a similar inconsistency is involved in standard $3N$ calculations employing the TM TBF.

The contributions of the TBF generated by the πN models 1, 2 and 2' of [14] to the $3N$ binding energy are listed in Table 1. These results are based on the separable representations of the original interaction models described in the preceeding section. Note that the value of 930 MeV is used for the expansion energy since we expect the average πN energy in the three-nucleon system to be near this value. In Table 1 also the result for model PJ of [2] is listed for the sake of comparison. Thus, it is easy to see that our results are qualitatively very similar to those of SA. Again the contibutions of the individual πN partial

Table 1. The contribution of the $\pi\pi$-exchange three-body force to the triton binding energy in keV for the πN models 1, 2 and 2' of the Jülich group [12,14] and PJ of [2]. The πNN form factor $F_{\pi NN}(q^2 = 0)$ determined consistenly from these models is given in the second line

	model 1'	model 2	model 2'	PJ
$F_{\pi NN}(0)$	0.934	0.925	0.962	0.811
S_{11}	-3.8	-2.8	-2.3	-4.8
S_{31}	35.1	64.4	51.7	26.4
P_{11}	-11.8	-26.4	-21.3	-8.8
P_{31}	1.8	9.0	6.4	-3.6
P_{13}	-2.2	-0.9	-1.1	4.5
P_{33}	-17.1	-31.6	-24.2	-16.0
$Total$	2.0	11.7	9.2	-2.3

waves to the $3N$ binding energy are very small (only in the order of keV) and there is again a strong cancellation between the attractive P_{11} and P_{33} partial waves and the repulsive S_{31} partial wave.

In [7] Murphy and Coon have presented a thorough comparison of the πN amplitudes used in the TM and Brazilian TBF and the one applied in the calculations by SA. Their main conclusion was that the two-orders-of-magnitude smaller value for the binding energy resulting from the $\pi\pi$-exchange TBF based upon the separable models of SA is most likely due to the very soft form factor resulting from these potentials. Therefore we want to look now at the πN form factors extracted from the models applied in the present study. The values for $F_{\pi NN}(0)$ are also given in Table 1. Note that model 2' yields the hardest πNN form factor which corresponds to a monopole form factor with a cutoff mass of about 708 MeV whereas the model PJ of SA corresponds to a monopole form factor with a cutoff mass of just 317 MeV. From comparing different columns of Table 1 one might conclude that there is some influence of the form factor on the magnitude of the TBF. However, the variations in the individual partial wave contributions are only around a factor of two or three and definitely not two orders of magnitude. Furthermore, the cancellation effects are independent of the softness of the πNN form factor and they tend to reduce the variations in the total contribution. In fact, the results for the πN models considered in the present paper lie all within a range of 15 keV, as can be seen from Table 1.

At this point one may wonder how reliable results based on a rank 1 separable approximation of the Jülich models are. In order to estimate the uncertainty due to the simplicity of the representation we constructed another rank 1 represention where the expansion energy was chosen at πN threshold (for the S-waves) or slightly above. Specifically we chose 1077 MeV for S_{11}, 1000 MeV for S_{31} and 1100 MeV for the other partial waves. The results for these alternative separable representations are compared with the ones obtained for our 'standard' choice in Table 2. From this Table we see that there are variations of the order of $20 \sim 30\%$ in some partial waves - but qualitatively there is no change in the results. Therefore we are confident that the used rank-1 separable representations are sufficiently accurate for the aim of the present investigation.

Table 2. The effect of choosing different expansion energies for the rank-1 separable representation in the EST expansion. Columns labelled with $E_i < E_{th}$ correspond to our standard choice of $E_i = 930$ MeV. The results given in the columns labelled with $E_i > E_{th}$ are obtained for the value of 1077 MeV for S_{11}, 1000 MeV for S_{31} and 1100 MeV for the other partial waves. All binding energies are given in keV

	model 1		model 2'	
	$E_i > E_{th}$	$E_i < E_{th}$	$E_i > E_{th}$	$E_i < E_{th}$
S_{11}	-5.4	-3.8	-5.5	-2.3
S_{31}	37.5	35.1	59.0	51.7
P_{11}	-11.8	-11.8	-20.6	-21.3
P_{31}	-1.0	1.8	0.3	6.4
P_{13}	-2.1	-2.2	-1.5	-1.1
P_{33}	-14.7	-17.1	-20.5	-24.2
Total	2.5	2.0	11.2	9.2

5 Discussion

We conclude from the previous section that the softness of the πNN form factor is not responsible for the smallness of the $\pi\pi$-exchange TBF based on πN potential models. If so, what makes the contribution to the three-body binding energy so small? In order to shed some light on this let us examine the basic two differences between the TM TBF and the one derived from a πN potential model. The first difference concerns the energy dependence. The original πN amplitude on which the TM TBF is based, is given in a covariant form and therefore depends on the energy (of the pion). However, in order to make this TBF suitable for application in standard (non-relativistic) 3N calculations the πN amplitude is expanded in powers of k_π/m_N and only the lowest order terms are kept. As a consequene the initial energy dependence drops completely out of the resulting TBF. In the approach of SA the TBF is energy dependent in a two-fold way, namely via the πN amplitude but also via the πNN form factor. The effect of switching off this energy dependence in the TBF has been analyzed thoroughly in [1,2] where it was found that it leads to a sizeable increase in the resulting binding energy. However, it was concluded by SA that the approximation of fixing the energy in the πN amplitude and the πNN form factor does not lead to a sufficiently large change in the contributions to make them comparable with the result for the TM TBF. None the less we would like to look at this point again because (unlike the potentials employed by SA) now the πN interaction model itself is energy dependent and therefore the effects from the energy dependence might be stronger. Note that in the present case the energy dependence of the TBF enters at three levels: (a) the energy dependence of the πN potential itself; (b) the energy dependence of the πN t-matrix; (c) the energy dependence of the πNN vertex function.

Table 3. The effect of removing the energy dependence in the πN potential (i), and in addition in the πNN vertex function and the πN ampliude (ii). All results are obtained by using model 2'. The binding energies are in *keV*

	exact	(i)	(ii)
S_{11}	-2.3	-3.4	-3.9
S_{31}	51.7	76.5	74.7
P_{11}	-21.3	-33.1	-39.9
P_{31}	6.4	13.7	14.4
P_{13}	-1.1	0.9	1.1
P_{33}	-24.2	-41.9	-45.7
Total	9.2	12.7	-0.7

We investigate the role of the energy dependence by fixing the energy at $E = 930$ MeV. This is the value chosen for constructing the separable interaction via the EST method and, accordingly, where the πN amplitudes generated by the Jülich model and its separable representation agree almost exactly. Thus, the results for our rank-1 separable interaction should be practically identical to the one for the original Jülich model in the particular case where the whole energy dependence (a)-(c) is fixed. Corresponding values are given in the last column of Table 3. The numbers in the third column of Table 3 are obtained by fixing only the energy dependence of the πN interaction. For the ease of comparison results without restricting the energy dependence are also shown in the Table. We see that the contribution of each πN partial wave is enhanced by about 50% after fixing the energy dependence of the potential. Fixing the energy dependence also in the πN t-matrix and the πNN form factor leads to a further increase in the contributions from the P_{11} and P_{33} waves and to a slight suppression in the S_{31}. However, those approximations definitely do not provide any really substantial enhancement of the resulting triton binding - in line with the findings of SA. All results shown in Table 3 are obtained by using model 2'. The other models behave qualitatively very similar and therefore we don't give the corresponding numbers here.

The other major difference between the TM force and the TBF based on a πN potential concerns the off-shell extrapolation of the πN amplitude. In the Jülich model the off-shell properties of the πN amplitude are completely determined by the dynamical ingredients of the πN model and the fit to the πN data. Since also for the potential models employed by SA the off-shell properties are, in principle, constrained by a fit to πN data let us emphasize the main difference here. In case of phenomenological separable interactions the off-shell behavior is, to a large extend, determined by the specific choice of the form-factor function. Moreover, separable interactions act only in single partial waves and accordingly the free parameters are determined by fitting only a single partial wave. On the other hand, in a meson-exchange potential like the Jülich model the dynamical ingredients give, in general, contributions to all partial waves and therefore the free parameters in this model - which are essentially the cutoff masses in the (baryon-baryon-meson) vertex form factors [12] - are much better constrained by a fit to the πN data. Clearly also here differences in the dynamics and/or differences in the parametrization of the vertex form factors will lead to variations in the off-shell properties of the resulting πN amplitude. Indeed such differences exist between the models 1 and 2 (or 2') considered here. But, as we have already seen in the last section, they to not lead to any significant variations in the results for the TBF.

The πN amplitude used in the TM force is given by (cf., e.g., [7,25])

$$\begin{aligned} T^{ij}_{\pi N}(\mathbf{k}_\pi, \mathbf{k}'_\pi) = F_{\pi NN}(\mathbf{k}^2_\pi) F_{\pi NN}(\mathbf{k}'^2_\pi) \\ \cdot \{\delta^{ij}[a + b\ \mathbf{k}_\pi \cdot \mathbf{k}'_\pi + c\ (\mathbf{k}^2_\pi + \mathbf{k}'^2_\pi)] \\ -d\ \epsilon^{ijk}\tau^k \boldsymbol{\sigma} \cdot \mathbf{k}_\pi \times \mathbf{k}'_\pi\}\ , \end{aligned} \tag{17}$$

where i, j are pion (cartesian) indices, $\mathbf{k}_\pi$ and $\mathbf{k}'_\pi$ are the momenta of the incoming and outgoing (off-shell) pions, and a, b, c, and d are constants defined, e.g., in [25]. Evidently the off-shell extrapolation is provided by the form factor function $F_{\pi NN}$ [24]. In the standard version of the TM force this form factors are assumed to be of monopole type,

$$F_{\pi NN}(\mathbf{k}^2_\pi) = \frac{\Lambda^2_\pi - m^2_\pi}{\Lambda^2_\pi + \mathbf{k}^2_\pi}, \tag{18}$$

with a cutoff mass $\Lambda_\pi = 5.8 m_\pi \approx 800$ MeV. We would like to emphasize at this point that, in principle, there is no connection between the πNN vertex (with an off-shell pion) and the πN amplitude entering into the TBF. It would appear only in the contribution of the direct nucleon pole diagram to the πN amplitude, which, however, is omitted in order to avoid double counting (cf. the discussion in Sect. 2). Therefore the prescription for the off-shell extrapolation used in the TM force must be considered as rather arbitrary.

In the discussion above we have tacitly ignored a conceptional subtlety when we talk about "off-shell". In case of the Jülich model the πN amplitude is obtained off-energy-shell, while an off-mass-shell πN amplitude is used in the TM potential. This means, that we can not make a simple comparison between them. The off-pion-mass-shell πN amplitude of the TM force depends on the pion momenta $\mathbf{k}_\pi$ and $\mathbf{k}'_\pi$. The off-energy-shell value of the πN amplitude for the potential model (at a certain energy) is given as a function of the relative momentum between the pion and the nucleon, $\mathbf{Q}$. Within non-relativistic kinematics $\mathbf{Q}$ is obtained by (cf. (10))

$$\mathbf{Q} = \frac{\mathbf{k}_\pi - \frac{m_\pi}{m_N}\mathbf{k}_N}{1 + \frac{m_\pi}{m_N}}, \tag{19}$$

where $\mathbf{k_N}$ is the nucleon momentum. One can see from this relation that $\mathbf{Q}$ becomes equivalent to $\mathbf{k}_\pi$ only in the limit of $m_\pi/m_N \to 0$.

In the following we want to discuss the off-shell properties entering into the calculations with the TM force and into the results presented in this paper. In view of the aforementioned difficulties it should be clear that any

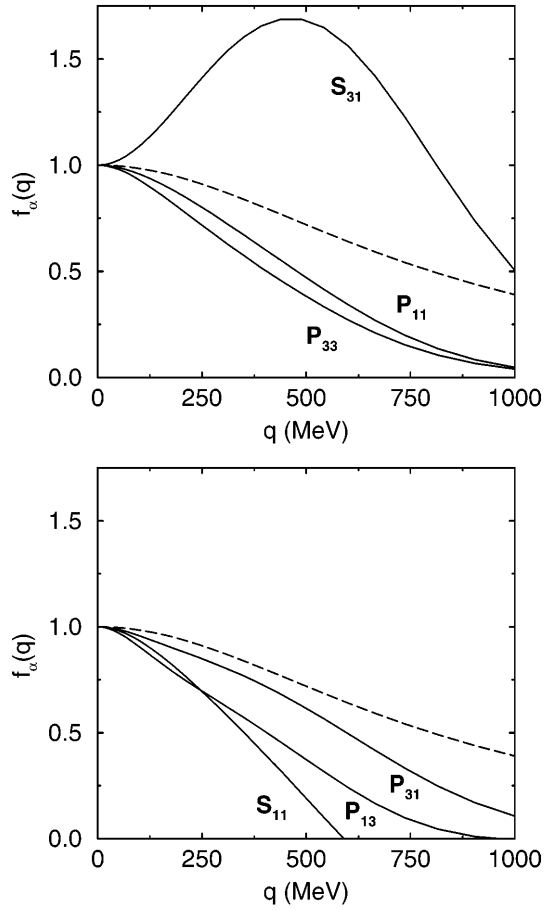

Fig. 8. Comparison of the monopole type function $F(q) = \Lambda^2/(\Lambda^2 + q^2)$ with $\Lambda = 800$ MeV with the off-energy shell behaviour of our model 2' for various πN partial waves. $f_\alpha(q)$ is defined in (20)

comparision can be only of qualitative nature. None the less, as we will see below such an analysis is useful and we believe that it indicates the source of the large discrepancy found in the contributions to the binding energy from the two approaches.

For this purpose let us introduce a half-off-shell function in the following way,

$$f_\alpha(Q) = \frac{T_\alpha(Q, Q'; Z)}{Q^l} \tag{20}$$

where $T_\alpha(Q, Q'; Z)$ is the off-shell πN t-matrix (projected on the partial wave α with angular momentum l) at a fixed energy Z and a fixed momentum Q'. The factor Q^l is taken out for convenience because then we can normalize these half-off-shell functions to 1 at $Q = 0$ for s- as well p-waves and we can easily compare them with each other. Corresponding results for the Jülich πN potential 2' are shown in Fig. 8, where Z and Q' in (20) have been fixed to 930 MeV and 130 MeV/c, respectively.

Let us first take a look at the p-waves and in particular at the P_{11} and P_{33} partial waves which provide the main attractive contributions to the TBF (cf. Table 1). In this case the momentum dependence of the TM πN amplitudes is roughly given by a monopole type function $F(\mathbf{Q}^2) = \Lambda^2/(\Lambda^2 + \mathbf{Q}^2)$ with $\Lambda = 800$ MeV, cf. (17-18), which is shown by the dashed curve in Fig. 8. We observe that the corresponding half-off-shell functions of the Jülich πN model fall off much faster with increasing (off-shell) momentum than this function. Accordingly we expect that a TBF based on the potential model will yield a much smaller attractive contribution to the three-body binding than one with off-shell properties similar to the monopole type function.

In case of the s-waves we get large repulsive contributions from the S_{31} partial waves and small attractive contributions from S_{11} (cf. Table 1). Please note that also the corresponding half-off-shell functions are radically different. The one for S_{31} exhibits a strong enhancement whereas the one for S_{11} falls off very strongly with increasing (off-shell) momentum. The s-wave part of the TM force (a- and c-terms), on the other hand, has no isospin dependence, cf. (17). This means that here the S_{31} and S_{11} partial waves have exactly the same momentum dependence. Therefore we suspect that a strong cancellation between the contributions from those two s-waves takes place. Indeed, actual triton calculations employing the TM force confirm that the total s-wave contributions are comparably small [26]. Evidently, such a cancellation does not occur with the TBF based on the Jülich πN model because of the differences in the off-shell properties. As a consequence, the (large) repulsive contribution of the S_{31} partial wave to the three-body binding survives (cf. Table 1).

Summarizing this phenomenological discussion of the off-shell properties we expect that a calculation based on the off-shell extrapolation used in the TM force should lead to a strong enhancement of the attractive contributions and at the same time reduce the repulsive contributions. We believe that this is the basic mechanism which makes the binding energy obtained with the TM TBF so large. We would like to substantiate this claim quantitatively with a model calculation. We can do this by substituting the off-shell properties of our πN model by the ones used in the TM force. This can be easily done for the separable representations that we are using. We only need to replace the Jülich off-shell behavior by defining the form factor of the separable potential, $g_\alpha(Q) := \langle Q|V_\alpha|\psi_{E_1}\rangle$, as follows:

$$g_\alpha(Q) \to \left(\lim_{Q\to 0} \frac{g_\alpha(Q)}{Q^l}\right) \frac{Q^l}{1 + Q^2/\Lambda^2} \ . \tag{21}$$

Furthermore we fix the energy dependence in the πN amplitude and the πNN vertex function again.

We demonstrate the effect on the binding energy for several values of Λ in Table 4. From this Table it is clear that we can get a substantial increase in the triton binding energy by choosing $\Lambda \approx 600 \sim 800$ MeV. Specifically one can see that the repulsion of the S_{31} partial wave is suppressed and, moreover, cancels to a large extent with the S_{11}. At the same time the attraction provided by the P_{11} and P_{33} partial waves is strongly enhanced - as we expected from analyzing Fig. 8. In fact, if we take into

Table 4. The effect of replacing the off-shell properties of our πN amplitude by monopole type functions (cf. 21) with different cutoff masses Λ. Starting point is model 2'. The binding energies are in keV

$\Lambda(MeV)$	200	400	600	800
S_{11}	-0.7	-4.8	-8.6	-11.0
S_{31}	0.4	7.0	13.4	17.5
P_{11}	-7.1	-28.5	-63.6	-100.2
P_{31}	-2.1	3.2	12.4	19.6
P_{13}	-1.1	1.1	5.6	9.3
P_{33}	-0.5	-48.7	-129.7	-202.7
$Total$	-11.1	-70.7	-170.5	-267.5

account that our results are based on first-order perturbation theory and therefore may underestimate the correct values by a factor two or even three [25] then our simulation with the choice $\Lambda = 800$ MeV practically reproduces the TM result, which is likewise based on $\Lambda_\pi = 800$ MeV.

6 Summary

In the present paper we have re-investigated the origin of the large discrepancy in the contribution of a $\pi\pi$-exchange TBF found by Saito and Afnan to the commenly accepted values obtained with the Tucson-Melbourne or Brazil TBF. Unlike SA, who employed phenomenological separable potentials, we started out from a meson-theoretical πN interaction model developed recently by the Jülich group. This model provides a good description of elastic πN scattering data. It is also in agreement with empirical information on the πN amplitude in the subthreshold region. In particular, it predicts the πN Σ term close to the empirical value. Furthermore, the decrease in the πNN form factors $F_{\pi NN}(q^2)$ from $q^2 = m_\pi^2$ to $q^2 = 0$ of about 4-7% and is comparable to that of the TM potential. Thus the form factors are much harder than those used by SA (which show a decrease of up to 20 %). Accordingly, the Jülich πN model does not show the deficiencies which, so far, have been thought to be the main reason for the small contribution of the $\pi\pi$-exchange TBF in SA's work.

None the less it turned out that also the $\pi\pi$-exchange TBF based on the πN amplitude of the Jülich model is very small. The contributions to the triton binding energy are in the order of a few keV, which means comparable to the results obtained by Saito and Afnan.

A detailed analysis of the main differences between the TM TBF and that derived from the Jülich πN potential model suggests that the differences in the contribution of the TBF to the 3N binding energy is due to the off-shell behaviour of the non-pole πN amplitude. In the Jülich model the off-shell properties of the πN amplitude are determined by the dynamical ingredients of the model and the fit to the πN data. As a result, the off-shell properties of the amplitude are different in the individual πN partial waves. In particular, the πN amplitudes that provide attractive contributions to the three-body binding (P_{11}, P_{33}) fall off relatively fast with increasing off-shell momentum while the repulsive S_{31} partial wave is enhanced. As a consequence, the total contribution of the TBF is very small as a result of the cancellation effects. On the other hand, in the πN amplitude underlying the TM TBF, the off-shell extrapolation is done in terms of a monopole form factor with a cut-off mass of 800 MeV for all partial waves. Since this monopole form factor falls off much slower then the P_{11} and P_{33} amplitudes of the Jülich πN model, the corresponding attraction provided by the TM force is considerably enhanced. At the same time, the repulsion in the S_{31} partial wave (and therefore any cancellation effects) is strongly suppressed. These combined effects do indeed explain the two-orders-of-magnitude discrepancy in the resulting contribution to the triton binding energy as we have demonstrated in a numerical model study.

Naturally the question arises how realistic and well-defined the off-shell properties of the Jülich πN model are, especially in view of the so-called quasi-potential ambiguity [27]. This is a topic which needs to be further investigated in the future, but it is certainly beyond the scope of the present paper. Here we only want to point to the fact that rather different ansatzes for the πN interaction (meson-exchange and simple separable forms, respectively) lead to qualitatively similar off-shell features and, in consequence, to similar results for the TBF. Qualitatively similar off-shell properties seem to be also predicted by other πN models - at least as far as we can judge from corresponding publications [28,29].

With regard to the off-shell extrapolation used in the πN amplitude on which the TM TBF is based the situation is, in our opinion, much less clear. First of all, we do not see any stringent physical reason for adopting the πNN form factor for this purpose, especially because no πNN vertex is present at all in the non-nucleon-pole part of the πN amplitude that enters into the derivation of the TBF. Furthermore, the choice of having the same off-shell properties in all πN partial waves is also hard to justify. Therefore the large contribution of the TM (and also the Brazil) TBF to the triton binding energy of around 1 MeV - though certainly desired by phenomenology - must be interpreted with caution.

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Appendix: Separable expansion of a potential with two terms

We consider to represent a potential V which consists of two terms, $V = V_1 + V_2$, in separable form. At first we expand V_2 by means of the standard EST-method, cf. (2)

to (6). Then the t-matrix $\tilde{T}_2(E)$ obtained from the separable representation $\tilde{V}_2$ agrees (on- as well as half-off-shell) with $T_2(E)$ corresponding to V_2 at the choosen expansion energies $E = E_i$, $i = 1, ..., N$.

Next we expand the potential V by assuming the following form

$$\tilde{V}(E) = |\tilde{v}_0\rangle \lambda_1(E) \langle \tilde{v}_0| + \tilde{V}_2. \tag{22}$$

We determine the "form factor" $|\tilde{v}_0\rangle$ in such a way that $\tilde{V}$ satisfies

$$V|\psi_\varepsilon\rangle = \tilde{V}(\varepsilon)|\psi_\varepsilon\rangle, \tag{23}$$

where $|\psi_\varepsilon\rangle$ is a solution of the scattering equation,

$$|\psi_\varepsilon\rangle = |k_\varepsilon\rangle + G_0(\varepsilon) V |\psi_\varepsilon\rangle, \tag{24}$$

at a fixed predetermined energy ε. We want to emphasize that we may choose the energy ε at which the potential V is expanded to be different from any of the energies E_i chosen for the separable expansion of V_2 . We see easily that (23) is satisfied if we choose $|\tilde{v}_0\rangle$ to be

$$|\tilde{v}_0\rangle = (V - \tilde{V}_2)\,|\psi_\varepsilon\rangle, \tag{25}$$

and $\lambda_1(\varepsilon)$ to be

$$\lambda_1(\varepsilon) = \frac{1}{\langle \psi_\varepsilon | V - \tilde{V}_2 | \psi_\varepsilon \rangle} \tag{26}$$

$$= \frac{1}{\langle \tilde{v}_0 | \psi_\varepsilon \rangle} \tag{27}$$

Note that the (half-off-shell) t-matrix $\tilde{T}(E)$ obtained from $\tilde{V}(\varepsilon)$ is identical to $T(E)$ obtained from V at $E = \varepsilon$ because of condition (23).

If $T(E)$ has a pole at E_p we may choose the expansion energy ε for the separable representation to be $\varepsilon = E_p$. The wave function at the pole, $|\psi_p\rangle$, is a solution of the equation

$$|\psi_p\rangle = G_0(E_p) V\,|\psi_p\rangle\ . \tag{28}$$

The t-matrix $\tilde{T}(E)$ for the separable potential $\tilde{V}(E)$ is given by

$$\tilde{T}(E) = |\tilde{v}(E)\rangle \frac{1}{1/\lambda_1(E) - \tilde{\Sigma}(E)} \langle \tilde{v}(E)| + \tilde{T}_2(E)\ , \tag{29}$$

where

$$\tilde{\Sigma}(E) = \langle \tilde{v}_0\,|\,G_0(E)\,|\,\tilde{v}(E)\rangle, \tag{30}$$

and

$$|\tilde{v}(E)\rangle = (1 + \tilde{T}_2(E) G_0(E))\,|\tilde{v}_0\rangle\ . \tag{31}$$

The "form factor" $|\tilde{v}_0\rangle$ is defined by

$$|\tilde{v}_0\rangle = (V - \tilde{V}_2)\,|\psi_p\rangle. \tag{32}$$

Substituting (31) and (32) into (30) leads to

$$\tilde{\Sigma}(E) = \langle \tilde{v}_0 | (1 + G_0(E)\tilde{T}_2(E)) G_0(E) V - G_0(E)\tilde{T}_2(E) | \psi_p \rangle. \tag{33}$$

We can evaluate $\tilde{\Sigma}(E)$ at $E = E_p$ by using (28),

$$\tilde{\Sigma}(E_p) = \langle \tilde{f}_0 | \psi_p \rangle = \frac{1}{\lambda_1(E_p)}\ , \tag{34}$$

where we have utilized, in addition, (27). By substituting this result into (29), we see that the t-matrix $\tilde{T}(E)$ obtained for the separable potential $\tilde{V}$ has the same pole position as the t-matrix of the original potential V.

Let us now assume that the first term of the original potential V has a separable form,

$$V_1(E) = |v_0\rangle \Lambda(E) \langle v_0|, \tag{35}$$

where $\Lambda(E)$ is

$$\Lambda(E) = \frac{1}{E - m_0}. \tag{36}$$

This is exactly the case for the (direct) nucleon pole contribution, Fig. 2(a). Then $|v_0\rangle$ corresponds to the bare πNN vertex function and m_0 is the bare nucleon mass. The solution of the Lippmann-Schwinger equation for $V = V_1 + V_2$ can then be written as

$$T(E) = |v(E)\rangle \frac{1}{1/\Lambda(E) - \Sigma(E)} \langle v(E)| + T_2(E), \tag{37}$$

where $T_2(E)$ is a solution of the scattering equation for the potential V_2. The self-energy $\Sigma(E)$ is given by

$$\Sigma(E) = \langle v_0 | G_0(E) | v(E) \rangle, \tag{38}$$

and the dressed πNN vertex function, $|v(E)\rangle$, by

$$|v(E)\rangle = (1 + T_2(E) G_0(E))\,|v_0\rangle. \tag{39}$$

Assuming that the t-matrix of (37) has a pole at $E = m_N$, $\Sigma(m_N)$ is evaluated as

$$\Sigma(m_N) = \frac{1}{\Lambda(m_N)} = m_N - m_0\ . \tag{40}$$

Also, the pole part of the t-matrix can be written in the form,

$$T(E) = |v(E)\rangle \frac{1}{1/\Lambda(E) - \Sigma(E)} \langle v(E)| = |v^R(E)\rangle \frac{1}{E - m_N} \langle v^R(E)|\ , \tag{41}$$

which defines the renormalized πNN vertex function $|v^R(E)\rangle$. At the pole $|v^R(E)\rangle$ is given by (cf., e.g., [2])

$$|v^R(m_N)\rangle = \frac{|v(m_N)\rangle}{(1 - \Sigma'(m_N))^{1/2}}, \tag{42}$$

where

$$\Sigma'(m_N) = -\langle v(m_N) | G_0^2(m_N) | v(m_N) \rangle\ . \tag{43}$$

In the last step we have assumed that V_2 does not dependent on the energy E. If it does (like in our case) then Σ' is determined by

$$\Sigma'(m_N) = \frac{\partial}{\partial E}\langle v_0|G_0(E)|v(E)\rangle \mid_{E=m_N} . \qquad (44)$$

Furthermore, we note that the renormalized vertex function is related to the πNN coupling constant $f_{\pi NN}$ and the πNN form factor $F_{\pi NN}$ by

$$f_{\pi NN}F_{\pi NN}(\mathbf{q}^2)a(q) \;=\; \langle q|v^R(m_N)\rangle \; . \qquad (45)$$

where $a(q)$ is a kinematical factor depending on the particular (ps or pv) πNN coupling (cf. (3) and (16) of [14]).

The wave function at the pole energy is obtained by solving (28)

$$|\psi_p\rangle = G_0(m_N)\,|v(m_N)\rangle. \qquad (46)$$

By substituing (46) into the lhs of (28), we get another relation between $|v(m_N)\rangle$ and $|\psi_p\rangle$,

$$|v(m_N)\rangle = V(m_N)|\psi_p\rangle. \qquad (47)$$

Now we expand the potential model $V(E)$ using the method described above. Then the resulting t-matrix obtained from the separable representation has the same pole position as the original interaction model. We will examine whether also the πNN coupling constant and form factor determined from the separable representation agree with the ones of the original interaction.

At first, we derive a relation between $|\tilde{v}(m_N)\rangle$ and $|v(m_N)\rangle$. Using (31), (32), and finally (47) we obtain

$$\begin{aligned}|\tilde{v}(m_N)\rangle &= (1+\tilde{T}_2(m_N)G_0(m_N))(V(m_N)-\tilde{V}_2)|\psi_p\rangle \\ &= V(m_N)|\psi_p\rangle \\ &= |v(m_N)\rangle. \end{aligned} \qquad (48)$$

(48) implies that the momentum dependence of the πNN vertex function obtained from the separable representation is identical to the one of the original potential.

The renormalized πNN vertex function at the pole, $|\tilde{v}^R(m_N)\rangle$, for the separable representation $\tilde{V}$ can be calculated in the same way as for the original potential. It is obtained by placing tildes over all quantities in (42) and (43):

$$|\tilde{v}^R(m_N)\rangle = \frac{|\tilde{v}(m_N)\rangle}{(1-\tilde{\Sigma}'(m_N))^{1/2}}, \qquad (49)$$

$$\tilde{\Sigma}'(m_N) = -\langle\tilde{v}(m_N)|G_0^2(m_N)|\tilde{v}(m_N)\rangle. \qquad (50)$$

Because of (48) we get

$$\tilde{\Sigma}'(m_N) = \Sigma'(m_N) \; . \qquad (51)$$

Finally, by substituting (48) and (51) into (49) it follows that

$$|\tilde{v}^R(m_N)\rangle = |v^R(m_N)\rangle \; , \qquad (52)$$

and consequently that $\tilde{f}_{\pi NN} = f_{\pi NN}$ and $\tilde{F}_{\pi NN}(\mathbf{q}^2) = F_{\pi NN}(\mathbf{q}^2)$.

In the above discussion we have not specified the form of $\lambda_1(E)$, since we needed only the value of $\lambda_1(E)$ at the energy $E = m_N$ to get the correct half-off shell t-matrix. If we assume $\lambda_1(E)$ to be of the form $1/\lambda_1(E) = E - \tilde{m}_0$, then $\tilde{m}_0$ can be determined by (27) (or (34)) to be

$$\tilde{m}_0 = m_N - \langle\psi_p|V - \tilde{V}_2|\psi_p\rangle. \qquad (53)$$

Therefore, by following the outlined procedure it is possible to construct a separable representation where (i) the half-off-shell behavior of the non-pole part of the t-matrix (at selected energies), (ii) the pole position, (iii) the πNN form factor, and (iv) the value of the πNN coupling constant are the same as in the original interaction model.

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